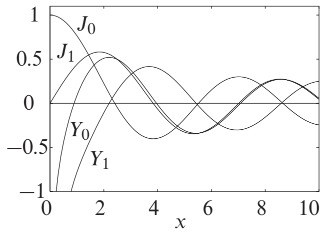
2.9 Special functions and polynomials

Gamma function

Definition	$\Gamma(z)=\int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad [\Re(z)>0]$	(2.407)
Relations	$n!=\Gamma(n+1)=n\Gamma(n) \quad (n=0,1,2,\ldots)$	(2.408)
	$\Gamma(1/2)=\pi^{1/2}$	(2.409)
	$\binom{z}{w}=\dfrac{z!}{w!(z-w)!}=\dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$	(2.410)
Stirling's formulas (for $\lvert z\rvert,n\gg 1$)	$\Gamma(z)\simeq \mathrm{e}^{-z}z^{z-(1/2)}(2\pi)^{1/2}\left(1+\dfrac{1}{12z}+\dfrac{1}{288z^2}-\cdots\right)$	(2.411)
	$n!\simeq n^{n+(1/2)}\mathrm{e}^{-n}(2\pi)^{1/2}$	(2.412)
	$\ln(n!)\simeq n\ln n-n$	(2.413)

Bessel functions

Series expansion	$J_\nu(x)=\left(\dfrac{x}{2}\right)^{\nu}\sum_{k=0}^{\infty}\dfrac{(-x^2/4)^k}{k!\Gamma(\nu+k+1)}$	(2.414)	$J_\nu(x)$	Bessel function of the first kind
	$Y_\nu(x)=\dfrac{J_\nu(x)\cos(\pi\nu)-J_{-\nu}(x)}{\sin(\pi\nu)}$	(2.415)	$Y_\nu(x)$	Bessel function of the second kind
			$\Gamma(\nu)$	Gamma function
			ν	order ($\nu\geq 0$)
Approximations	$J_\nu(x)\simeq\begin{cases}\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu} & (0\leq x\ll\nu)\\ \left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x-\frac{1}{2}\nu\pi-\frac{\pi}{4}\right) & (x\gg\nu)\end{cases}$	(2.416)		
	$Y_\nu(x)\simeq\begin{cases}\frac{-\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu} & (0< x\ll\nu)\\ \left(\frac{2}{\pi x}\right)^{1/2}\sin\left(x-\frac{1}{2}\nu\pi-\frac{\pi}{4}\right) & (x\gg\nu)\end{cases}$	(2.417)		
Modified Bessel functions	$I_\nu(x)=(-\mathbf{i})^{\nu}J_\nu(\mathbf{i}x)$	(2.418)	$I_\nu(x)$	modified Bessel function of the first kind
	$K_\nu(x)=\dfrac{\pi}{2}\mathbf{i}^{\nu+1}[J_\nu(\mathbf{i}x)+\mathbf{i}Y_\nu(\mathbf{i}x)]$	(2.419)	$K_\nu(x)$	modified Bessel function of the second kind
Spherical Bessel function	$j_\nu(x)=\left(\dfrac{\pi}{2x}\right)^{1/2}J_{\nu+\frac{1}{2}}(x)$	(2.420)	$j_\nu(x)$	spherical Bessel function of the first kind [similarly for $y_\nu(x)$]



Legendre polynomials[a]

Legendre equation	$$(1-x^2)\frac{d^2P_l(x)}{dx^2}-2x\frac{dP_l(x)}{dx}+l(l+1)P_l(x)=0 \quad (2.421)$$	P_l Legendre polynomials l order ($l \geq 0$)
Rodrigues' formula	$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \quad (2.422)$$	
Recurrence relation	$$(l+1)P_{l+1}(x)=(2l+1)xP_l(x)-lP_{l-1}(x) \quad (2.423)$$	
Orthogonality	$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx=\frac{2}{2l+1}\delta_{ll'} \quad (2.424)$$	$\delta_{ll'}$ Kronecker delta
Explicit form	$$P_l(x)=2^{-l}\sum_{m=0}^{l/2}(-1)^m\binom{l}{m}\binom{2l-2m}{l}x^{l-2m} \quad (2.425)$$	$\binom{l}{m}$ binomial coefficients
Expansion of plane wave	$$\exp(\mathbf{i}kz)=\exp(\mathbf{i}kr\cos\theta) \quad (2.426)$$ $$=\sum_{l=0}^{\infty}(2l+1)\mathbf{i}^l j_l(kr)P_l(\cos\theta) \quad (2.427)$$	k wavenumber z propagation axis $z=r\cos\theta$ j_l spherical Bessel function of the first kind (order l)

$P_0(x)=1$	$P_2(x)=(3x^2-1)/2$	$P_4(x)=(35x^4-30x^2+3)/8$
$P_1(x)=x$	$P_3(x)=(5x^3-3x)/2$	$P_5(x)=(63x^5-70x^3+15x)/8$

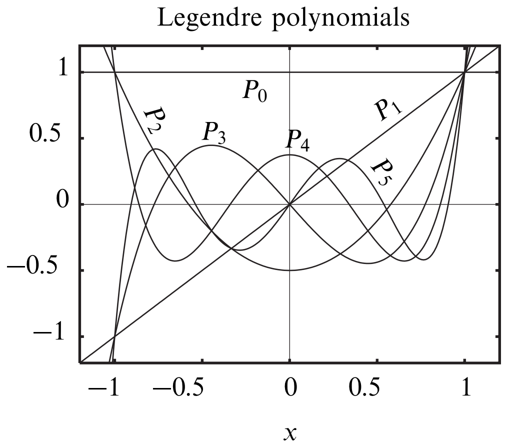
[a] Of the first kind.

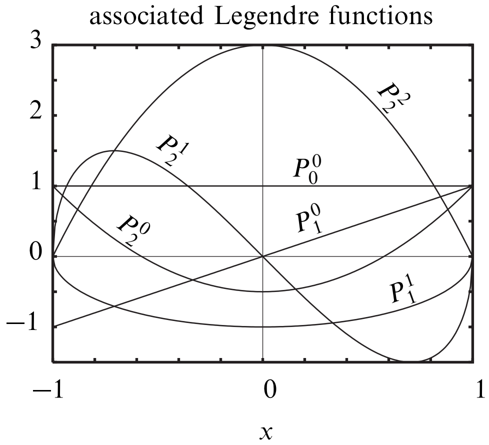
Associated Legendre functions[a]

Associated Legendre equation	$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m(x)}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P_l^m(x)=0 \quad (2.428)$$	P_l^m associated Legendre functions
From Legendre polynomials	$$P_l^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x), \quad 0\leq m\leq l \quad (2.429)$$ $$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \quad (2.430)$$	P_l Legendre polynomials
Recurrence relations	$$P_{m+1}^m(x)=x(2m+1)P_m^m(x) \quad (2.431)$$ $$P_m^m(x)=(-1)^m(2m-1)!!(1-x^2)^{m/2} \quad (2.432)$$ $$(l-m+1)P_{l+1}^m(x)=(2l+1)xP_l^m(x)-(l+m)P_{l-1}^m(x) \quad (2.433)$$	!! $5!!=5\cdot3\cdot1$ etc.
Orthogonality	$$\int_{-1}^{1}P_l^m(x)P_{l'}^m(x)\,dx=\frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{ll'} \quad (2.434)$$	$\delta_{ll'}$ Kronecker delta

$P_0^0(x)=1$	$P_1^0(x)=x$	$P_1^1(x)=-(1-x^2)^{1/2}$
$P_2^0(x)=(3x^2-1)/2$	$P_2^1(x)=-3x(1-x^2)^{1/2}$	$P_2^2(x)=3(1-x^2)$

[a] Of the first kind. $P_l^m(x)$ can be defined with a $(-1)^m$ factor in Equation (2.429) as well as Equation (2.430).





Spherical harmonics

Differential equation	$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_l^m+l(l+1)Y_l^m=0$ (2.435)	Y_l^m	spherical harmonics
Definition[a]	$Y_l^m(\theta,\phi)=(-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$ (2.436)	P_l^m	associated Legendre functions
Orthogonality	$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}Y_l^{m*}(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=\delta_{mm'}\delta_{ll'}$ (2.437)	Y^* $\delta_{ll'}$	complex conjugate Kronecker delta
Laplace series	$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}Y_l^m(\theta,\phi)$ (2.438) where $a_{lm}=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}Y_l^{m*}(\theta,\phi)f(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ (2.439)	f	continuous function
Solution to Laplace equation	if $\nabla^2\psi(r,\theta,\phi)=0$, then $\psi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_l^m(\theta,\phi)\cdot\left[a_{lm}r^l+b_{lm}r^{-(l+1)}\right]$ (2.440)	ψ a,b	continuous function constants

$$Y_0^0(\theta,\phi)=\sqrt{\frac{1}{4\pi}} \qquad Y_1^0(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^{\pm1}(\theta,\phi)=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\pm\mathrm{i}\phi} \qquad Y_2^0(\theta,\phi)=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$

$$Y_2^{\pm1}(\theta,\phi)=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,\mathrm{e}^{\pm\mathrm{i}\phi} \qquad Y_2^{\pm2}(\theta,\phi)=\sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathrm{e}^{\pm2\mathrm{i}\phi}$$

$$Y_3^0(\theta,\phi)=\frac{1}{2}\sqrt{\frac{7}{4\pi}}(5\cos^2\theta-3)\cos\theta \qquad Y_3^{\pm1}(\theta,\phi)=\mp\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta(5\cos^2\theta-1)\mathrm{e}^{\pm\mathrm{i}\phi}$$

$$\pm2(\theta,\phi)=\frac{1}{\ }\sqrt{\frac{105}{2\pi}}\ \ 2\ \ \pm2\mathrm{i}\phi \qquad \pm3\ \ \frac{1}{4}\sqrt{\frac{35}{4\pi}}\ \ 3\ \ \pm3\mathrm{i}\phi$$

Delta functions

Kronecker delta	$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$	(2.441)	δ_{ij} Kronecker delta $i,j,k,\ldots$ indices ($=1,2$ or 3)
	$\delta_{ii} = 3$	(2.442)	
Three-dimensional Levi–Civita symbol (permutation tensor)[a]	$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$ all other $\epsilon_{ijk} = 0$	(2.443)	ϵ_{ijk} Levi–Civita symbol (see also page 25)
	$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$	(2.444)	
	$\delta_{ij}\epsilon_{ijk} = 0$	(2.445)	
	$\epsilon_{ilm}\epsilon_{jlm} = 2\delta_{ij}$	(2.446)	
	$\epsilon_{ijk}\epsilon_{ijk} = 6$	(2.447)	
Dirac delta function	$\int_a^b \delta(x)\,\mathrm{d}x = \begin{cases} 1 & \text{if } a<0<b \\ 0 & \text{otherwise} \end{cases}$	(2.448)	$\delta(x)$ Dirac delta function $f(x)$ smooth function of x a,b constants
	$\int_a^b f(x)\delta(x-x_0)\,\mathrm{d}x = f(x_0)$	(2.449)	
	$\delta(x-x_0)f(x) = \delta(x-x_0)f(x_0)$	(2.450)	
	$\delta(-x) = \delta(x)$	(2.451)	
	$\delta(ax) = \lvert a\rvert^{-1}\delta(x) \quad (a \neq 0)$	(2.452)	
	$\delta(x) \simeq n\pi^{-1/2}\mathrm{e}^{-n^2x^2} \quad (n \gg 1)$	(2.453)	

[a]The general symbol $\epsilon_{ijk\ldots}$ is defined to be $+1$ for even permutations of the suffices, -1 for odd permutations, and 0 if a suffix is repeated. The sequence $(1,2,3,\ldots,n)$ is taken to be even. Swapping adjacent suffices an odd (or even) number of times gives an odd (or even) permutation.